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1990 J. Phys. A: Math. Gen. 23 5061

(http://iopscience.iop.org/0305-4470/23/21/040)

J. Phys. A: Math. Gen. 23 (1990) 5061-5081. Printed in the UK

Symmetries and constants of motion for constrained Lagrangian systems: a presymplectic version of the Noether theorem

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Received 24 May 1990

Abstract. The concept of dynamical symmetry for constrained Lagrangian systems is analysed and generalized. Both the direct and converse Noether theorems are proved for singular Lagrangians.

1. Introduction

One of the most significant results of the geometric-differential study of classical mechanics is without doubt a greater understanding and generalization of the Noether theorem [1-3]. The Noether theorem proves that each infinitesimal symmetry of the Lagrange function is a dynamical symmetry (i.e. it leaves the equations of motion unchanged) associated with a constant of motion. The generalized version of the theorem is based on the fact that dynamical symmetries may be Newtonoid, rather than simply Newtonian transformations [2]. Only by accepting such a generality is it possible to formulate a converse theorem associating a dynamical symmetry with each constant of motion. In this manner the results obtained with the Lagrangian formalism become equivalent to those of the Hamiltonian formalism.

In brief the results obtained in the above-mentioned works will be quoted here and in order to do so use will be made of the standard tools of the geometrical description of the dynamics on TQ, the tangent space of the configuration space Q. For definitions, notations and properties, the reader's attention is drawn to [4-6].

Therefore, in the present work S will be used to indicate the vertical endomorphism, Γ_0, any second-order vector field and Δ the Liouville field. One must, however, recall that $S(\Gamma_0)=\Delta$. As in [2] every vector field $X(\Gamma_0)\in\mathscr{X}(TQ)$ which satisfies:

$$S[X(\Gamma_0),\Gamma_0]=0 \tag{1.1}$$

will here be called a Newtonoid field, with respect to a second-order field Γ_0.

The infinitesimal (Newtonoid) transformation

$$X(\Gamma_0):=X+S[\Gamma_0,X] \tag{1.2}$$

can be associated with each vector field $X\in\mathscr{X}(TQ)$.

By definition, a symmetry of the Lagrangian $\mathscr{L}$ is a field $X\in\mathscr{X}(TQ)$ which, for a particular assigned function $F\in\mathscr{F}(TQ)$, satisfies

$$L_{X(\Gamma_0)}\mathscr{L}=L_{\Gamma_0}F \tag{1.3}$$

for each Γ_0 field of the second order.

Each Lagrange function is associated with a closed 2-form $\omega_{\mathscr{L}} = -\mathrm{d}(\mathrm{d}\mathscr{L} \circ S)$ which proves to be symplectic if $\mathscr{L}$ is regular; that is if

$$\det W = \det\left(\frac{\partial^2 \mathscr{L}}{\partial u\, \partial u}\right) \neq 0 \tag{1.4}$$

with $u = \mathrm{d}q/\mathrm{d}t$.

If (1.4) holds true, one, and only one, second-order dynamics $\Gamma \in \mathscr{X}(TQ)$ exists, which resolves the (algebraic) equation

$$i_\Gamma \omega_{\mathscr{L}} = \mathrm{d}E \tag{1.5}$$

where E is the Lagrangian energy

$$E := L_\Delta \mathscr{L} - \mathscr{L}. \tag{1.6}$$

Furthermore, equation (1.5) intrinsically expresses the Lagrange equations. Indeed, defining

$$G := i_{X(\Gamma_0)}(\mathrm{d}\mathscr{L} \circ S) - F \tag{1.7}$$

it was demonstrated in [2] that, if (1.3) holds true, this implies

$$L_\Gamma G = 0 \tag{1.8}$$

$$i_{X(\Gamma)} \omega_{\mathscr{L}} = \mathrm{d}G \tag{1.9}$$

$$L_{X(\Gamma)} E = 0 \tag{1.10}$$

$$[X(\Gamma), \Gamma] = 0. \tag{1.11}$$

On the other hand, the converse Noether theorem can be used to show that, if (1.8) holds true, (1.9), (1.10), (1.11) and (1.3) all hold true with F being defined by (1.7). From a geometrical point of view, in [1-3] the symplectic manifold structure, attributed to TQ through $\omega_{\mathscr{L}}$, plays an essential role in associating symmetries and constants of motion. If $\mathscr{L}$ is not regular and the rank of W is constant, TQ is said to be a presymplectic manifold [7]. Therefore, equation (1.5) no longer has a single solution and, in general, the motion is constrained on some submanifold of TQ as will be clarified here below. In this case the Noether theorem is to be studied in the light of the constraint theory [8-15]. Using this approach and starting from hypothesis (1.3) some authors [11] have obtained results similar to (1.8), (1.9) and (1.10) on constraint submanifolds. Still others [16-18] have also inferred dynamical symmetry properties by applying particular hypotheses to $X(\Gamma)$ or to the nature of the constraints.

In the present work it will be demonstrated that the direct and converse Noether theorems hold true even if hypothesis (1.4) is removed, therefore, setting aside the hypothesis that TQ is a symplectic manifold. In essence, to do so, the definition of invariance of the equations of motion will be adapted to the case of degenerate Lagrangians. Furthermore, it will be demonstrated that the infinitesimal transformation $X(\Gamma)$, connected to the constant of motion in both directions by the Noether theorem, is always tangent to the constraint submanifold and that it is a dynamical symmetry over it.

In the following section a brief summary will be given of the notations used in the theory of constraints and of the most significant results obtained. In section 3 some geometrical tools, useful in the analysis of motion on the submanifolds defined by the constraints, will be identified. Section 4 is dedicated to the definition of dynamical symmetries and to the demonstration of the direct Noether theorem. Finally, in section 5 the converse version of this theorem will be demonstrated starting from a function which is a constant of motion on the constraint submanifold and, from that point, building a corresponding dynamical symmetry.

The standard hypotheses used in the literature will be applied: the rank of W is constant on all TQ while that of the Poisson brackets of constraints is constant on the phase space T^*Q; the secondary Hamiltonian constraints cannot be conditions only on the coordinates of Q. Finally, without loss of generality, the assumption will be made that no tertiary Hamiltonian constraints exist.

2. Preliminary results

Let n be the number of degrees of freedom of a dynamical system. As is known, if the rank of the Hessian matrix (1.4) is $n-m$, the Legendre mapping $F\mathscr{L}$ (the fibre derivative of the Lagrangian) is not invertible. Therefore a number m of primary Hamiltonian constraints exists in T^*Q. Taking the notations used in [12], they will be here indicated by

$$\phi_\mu^{(0)}=0 \qquad \mu=1, m. \tag{2.1}$$

They define a submanifold $M_0 \subset T^*Q$. Each function $H \in \mathscr{F}(T^*Q)$ such that $F\mathscr{L}^*H=E$ is a Hamiltonian for the system. The time evolution of any $g \in \mathscr{F}(T^*Q)$ is given on M_0 by

$$\frac{\mathrm{d}g}{\mathrm{d}t} \underset{M_0}{=} \{g, H\}+\lambda^\mu\{g, \phi_\mu^{(0)}\} \tag{2.2}$$

with arbitrary $\lambda^\mu \in \mathscr{F}(T^*Q)$. The constraints $\phi_{\mu_0}^{(0)}$ with $\mu_0=1, m_1$ $(m_1 \leqslant m)$ such that

$$0 \underset{M_0}{=} \{\phi_{\mu_0}^{(0)}, \phi_\mu^{(0)}\} \qquad \mu_0=1, m_1;\ \mu=1, m \tag{2.3}$$

are first-class functions [8] on M_0. For the other constraints the following holds true

$$0 \underset{M_0}{\neq} \det|\{\phi_{\mu_0'}^{(0)}, \phi_{\nu_0'}^{(0)}\}| \qquad \mu_0', \nu_0'=1, m-m_1. \tag{2.4}$$

By requiring the conservation of the constraints, on the one hand, we obtain a determination of the multipliers $\lambda^{\mu_0'}$ which makes it possible to define

$$H^{(1)}=H+\lambda^{\mu_0'}\phi_{\mu_0'}^{(0)} \tag{2.5}$$

so that

$$0 \underset{M_0}{=} \{\phi_{\mu_0'}^{(0)}, H^{(1)}\} \qquad \mu_0'=1, m-m_1. \tag{2.6}$$

On the other hand, secondary constraints can be obtained

$$\phi_{\mu_0}^{(1)}=\{\phi_{\mu_0}^{(0)}, H^{(1)}\}=0 \qquad \mu_0=1, m_1 \tag{2.7}$$

which, together with the primary constraints, define the submanifold $M_1 \subset M_0$. Thus, from among $\phi_{\mu_0}^{(0)}$, it is possible to choose the functions $\phi_{\mu_1}^{(0)}$, with $\mu_1=1, m_2$ $(m_2 \leqslant m_1)$, which are first-class functions on M_1:

$$0 \underset{M_1}{=} \{\phi_{\mu_1}^{(0)}, \phi_{\mu_0}^{(1)}\} \qquad \mu_1=1, m_2;\ \mu_0=1, m_1 \tag{2.8}$$

$$0 \underset{M_1}{\neq} \det|\{\phi_{\mu_1'}^{(1)}, \phi_{\nu_1'}^{(0)}\}| \qquad \mu_1', \nu_1'=1, m_1-m_2. \tag{2.9}$$

Since, by hypothesis, M_1 is the final submanifold, by requiring conservation of (2.7) no new constraints are obtained; nevertheless the multipliers $\lambda^{\mu'_1}$ are determined on M_1 so that the Hamiltonian

$$H^{(2)} = H^{(1)} + \lambda^{\mu'_1}\phi^{(0)}_{\mu'_1} \tag{2.10}$$

is a first-class function on M_1. The multipliers λ^{μ_1} associated with the first-class primary constraints remain arbitrary and the dynamical evolution of a function $g \in \mathscr{F}(T^*Q)$ is finally given by

$$\frac{\mathrm{d}g}{\mathrm{d}t} \underset{M_1}{=} \{g, H^{(2)}\} + \lambda^{\mu_1}\{g, \phi^{(0)}_{\mu_1}\}. \tag{2.11}$$

On the other hand, the Lagrangian constraints arise because $\omega_{\mathscr{L}}$ is degenerate and therefore, in general, no second-order field exists satisfying (1.5) on all TQ. As is known [14], it is possible to associate such constraints with Hamiltonian constraints by means of the differential operator K, whose effect on each function $f \in \mathscr{F}(T^*Q)$ is given by

$$Kf = F\mathscr{L}^*\{f, H\} + v^\mu F\mathscr{L}^*\{f, \phi^{(0)}_\mu\} \tag{2.12}$$

where v^μ are known functions of $\mathscr{F}(TQ)$. The primary Lagrangian constraints, identifing the submanifold S_1 on which (1.5) has at least one second-order solution, can be obtained as

$$0 = \chi^{(1)}_\mu = K\phi^{(0)}_\mu \qquad \mu = 1, m. \tag{2.13}$$

Corresponding to the subdivision (2.3) and (2.4) of the primary Hamiltonian constraints, with the help of (2.6) and (2.7) the constraints (2.13) are split in the dynamical constraints

$$0 = \chi^{(1)}_{\mu_0} = F\mathscr{L}^*\phi^{(1)}_{\mu_0} \qquad \mu_0 = 1, m_1 \tag{2.14}$$

and in the so-called SODE (second-order differential equations) conditions

$$O = \chi^{(1)}_{\mu'_0} = (v^{\nu'_0} - F\mathscr{L}^*\lambda^{\nu'_0})F\mathscr{L}^*\{\phi^{(0)}_{\mu'_0}, \phi^{(0)}_{\nu'_0}\} \qquad \mu'_0 = 1, m - m_1. \tag{2.15}$$

The constraints (2.15) are not $F\mathscr{L}$ projectable, that is no function of $\mathscr{F}(T^*Q)$ exists whose pullback by $F\mathscr{L}$ yields $\chi^{(1)}_{\mu'_0}$. Since, for (2.4) the matrix $|\{\phi^{(0)}_{\mu'_0}, \phi^{(0)}_{\nu'_0}\}|$ is invertible the following will be used as an expression of the SODE constraints

$$v^{\mu'_0} - F\mathscr{L}^*\lambda^{\mu'_0} = 0 \qquad \mu'_0 = 1, m - m_1. \tag{2.16}$$

Requiring the field Γ which is the solution of (1.5) on S_1 also to be tangent to S_1 gives rise to secondary Lagrangian constraints

$$0 = \chi^{(2)}_{\mu_0} \underset{S_1}{=} K\phi^{(1)}_{\mu_0} \qquad \mu_0 = 1, m_1 \tag{2.17}$$

which, together with constraints (2.13) define the submanifold $S_2 \subset S_1 \subset TQ$. Since no tertiary Hamiltonian constraints exist (2.17) can be reduced to

$$0 = \chi^{(2)}_{\mu'_1} \underset{S_1}{=} (v^{\nu'_1} - F\mathscr{L}^*\lambda^{\nu'_1})F\mathscr{L}^*\{\phi^{(1)}_{\mu'_1}, \phi^{(0)}_{\nu'_1}\} \qquad \mu'_1 = 1, m_1 - m_2 \tag{2.18}$$

which are not $F\mathscr{L}$ projectable constraints and which, as was done to get (2.15), will be written as follows

$$v^{\mu_1'} - F\mathscr{L}^* \lambda^{\mu_1'} = 0 \qquad \mu_1' = 1, m_1 - m_2. \tag{2.19}$$

Another useful correlation between T^*Q and TQ is the correlation between primary Hamiltonian constraints and the vector fields of the set

$$\operatorname{Ker} \omega_{\mathscr{L}} := \{X \in \mathscr{X}(TQ) \backslash i_X \omega_{\mathscr{L}} = 0\}. \tag{2.20}$$

Indicating with $V(TQ)$ the set of the vertical vector fields

$$V(TQ) := \{X \in \mathscr{X}(TQ) \backslash S(X) = 0\} \tag{2.21}$$

it is easy to see [12] that a basis for

$$V(\operatorname{Ker} \omega_{\mathscr{L}}) := \operatorname{Ker} \omega_{\mathscr{L}} \cap V(TQ) \tag{2.22}$$

is made up of the fields

$$K^v_\mu = \left(F\mathscr{L}^* \frac{\partial \phi^{(0)}_\mu}{\partial p} \right) \frac{\partial}{\partial u} \qquad \mu = 1, m \tag{2.23}$$

while it is possible to choose the remaining fields within the basis of $\operatorname{Ker} \omega_{\mathscr{L}}$ as follows [19]

$$K_{\mu_0} = \left(F\mathscr{L}^* \frac{\partial \phi^{(0)}_{\mu_0}}{\partial p} \right) \frac{\partial}{\partial q} + \left[F\mathscr{L}^* \left\{ \frac{\partial H}{\partial p}, \phi^{(0)}_{\mu_0} \right\} \right.$$
$$\left. + v^\mu F\mathscr{L}^* \left\{ \frac{\partial \phi^{(0)}_\mu}{\partial p}, \phi^{(0)}_{\mu_0} \right\} \right] \frac{\partial}{\partial u} \qquad \mu_0 = 1, m_1. \tag{2.24}$$

The properties of these fields in respect to the known functions v^μ appearing in (2.12) are

$$L_{K^v_\mu} v^\nu = \delta^\nu_\mu \qquad \mu, \nu = 1, m \tag{2.25}$$

$$L_{K_{\mu_0}} v^\nu = 0 \qquad \mu_0 = 1, m_1; \ \nu = 1, m. \tag{2.26}$$

Since, as was shown in [11, 13], a necessary and sufficient condition for $f \in \mathscr{F}(TQ)$ to be $F\mathscr{L}$-projectable is that

$$L_{K^v_\mu} f = 0 \qquad \mu = 1, m \tag{2.27}$$

the properties (2.25) show that the functions v^μ are typically non-projectable.

From this point on we will indicate with Γ the second-order dynamics obtained upon completion of the analysis of consistence in TQ. The dynamics Γ satisfies (1.5) on S_1 and is tangent to S_2. It is worth recalling that such a dynamics is determined except for an arbitrary linear combination of elements of $V(\operatorname{Ker} \omega_{\mathscr{L}})$ which must be tangent to S_2. These elements are the vector fields $K^v_{\mu_1}$, with $\mu_1 = 1, m_2$, and they match the number of first-class primary Hamiltonian constraints on M_1. On the other hand, the fields $K^v_{\mu_0}$, with $\mu_0 = 1, m_1$, are tangent only to S_1 whereas all the fields of $V(\operatorname{Ker} \omega_{\mathscr{L}})$ are tangent to the submanifold defined by the dynamical constraints, because the latter are $F\mathscr{L}$ projectable.

Now we have all the elements required to study the case in which a singular Lagrangian presents a generalized symmetry, as in the case in (1.3). It must, first of all, be recalled that in the literature [11] it has proved that, for degenerate Lagrangian systems, the following results derive directly from (1.3):

$$L_{K_\mu^v} G = 0 \qquad \mu = 1, m \tag{2.28}$$

$$L_{K_{\mu_0}} G \underset{S_1}{=} 0 \qquad \mu_0 = 1, m_1 \tag{2.29}$$

$$(i_{X(\Gamma_0)}\omega_{\mathscr{L}} - \mathrm{d}G) \circ S = 0 \qquad \forall \Gamma_0 \colon S(\Gamma_0) = \Delta \tag{2.30}$$

$$i_{X(\Gamma)}\omega_{\mathscr{L}} - \mathrm{d}G \underset{S_1}{=} 0 \tag{2.31}$$

$$L_{X(\Gamma)} E \underset{S_1}{=} 0 \tag{2.32}$$

$$L_{\Gamma_0} G = i_{X(\Gamma_0)}(\mathrm{d}E - i_{\Gamma_0}\omega_{\mathscr{L}}) \qquad \forall \Gamma_0 \colon S(\Gamma_0) = \Delta \tag{2.33}$$

$$L_\Gamma G \underset{S_1}{=} 0. \tag{2.34}$$

The relations in (2.28) imply that a function $G_0 \in \mathscr{F}(T^*Q)$ exists such that

$$F\mathscr{L}^* G_0 = G \tag{2.35}$$

and therefore (2.29) and (2.34) respectively imply

$$\{G_0, \phi_{\mu_0}^{(0)}\} \underset{M_1}{=} 0 \qquad \mu_0 = 1, m_1 \tag{2.36}$$

$$\{G_0, H^{(1)}\} \underset{M_1}{=} 0. \tag{2.37}$$

Like G, G_0 is also a constant of motion.

To complement these results some considerations should be added which will be useful in the next section, comments which bring further properties of G_0 to light. It will be demonstrated that G_0 can always be chosen so as to be a first-class function on M_1. Since G_0 is not unambiguous we can write

$$G = F\mathscr{L}^* G_0 = F\mathscr{L}^*(G_0' + w^\mu \phi_\mu^{(0)}) \tag{2.38}$$

with arbitrary $w^\mu \in \mathscr{F}(T^*Q)$, and $G_0' \in \mathscr{F}(T^*Q)$ such that $G = F\mathscr{L}^* G_0'$. Once the function G_0' has been assigned, (2.4) ensures that the $m - m_1$ functions $w^{\mu_0'}$ can be determined so that

$$\{G_0, \phi_{\mu_0'}^{(0)}\} \underset{M_0}{=} 0 \qquad \mu_0' = 1, m - m_1. \tag{2.39}$$

Similarly, keeping (2.9) in mind, $m_1 - m_2$ multipliers $w^{\mu_1'}$ can be determined so that

$$\{G_0, \phi_{\mu_1'}^{(1)}\} \underset{M_1}{=} 0 \qquad \mu_1' = 1, m_1 - m_2. \tag{2.40}$$

From the Jacobi identity, together with (2.36) and (2.37), and keeping in mind the fact that the functions $\phi_{\mu_1}^{(0)}$ are first class, it follows that

$$\{G_0, \phi_{\mu_1}^{(1)}\} \underset{M_1}{=} \{\{G_0, \phi_{\mu_1}^{(0)}\}, H^{(1)}\} \underset{M_1}{=} \{\{G_0, \phi_{\mu_1}^{(0)}\}, H^{(2)}\} \underset{M_1}{=} 0 \tag{2.41}$$

where the right-hand side vanishes because no tertiary Hamiltonian constraints exist. Relations (2.36), (2.39), (2.40) and (2.41) demonstrate that G_0 is first class on M_1.

3. Weakly Hamiltonian vector fields

In this section some results are given which prove highly useful in checking the tangency of vector fields of $\mathscr{X}(TQ)$ to constraint submanifolds. Some remarks must first be made on the notations.

As is known, if P is a differential manifold and $S \subset P$ a submanifold, given X, $Y \in \mathscr{X}(P)$ and $f \in \mathscr{F}(P)$, if

$$X|_S = Y|_S \tag{3.1}$$

then

$$L_X f \underset{S}{=} L_Y f. \tag{3.2}$$

Replacing f in (3.2) with the constraints which define S, it is easy to see that X is tangent to S if and only if Y is tangent to S. This elementary property will often be used here below. In order to avoid particularly difficult notation, (3.1) will be written as

$$X \underset{S}{=} Y. \tag{3.3}$$

X_f will be used to indicate the Hamiltonian vector field belonging to $\mathscr{X}(T^*Q)$ and which is the solution of the (algebraic) equation

$$i_{X_f}\Omega = \mathrm{d}f \qquad f \in \mathscr{F}(T^*Q) \tag{3.4}$$

where Ω is the canonical symplectic 2-form.

Checking the tangency of Hamiltonian vector fields to some submanifold of T^*Q is not a problem, basically because $f \in \mathscr{F}(T^*Q)$ is first class on a submanifold if and only if its Hamiltonian field is tangent to said submanifold. If the 2-form $\omega_{\mathscr{L}} = F\mathscr{L}^*\Omega$ is degenerate it proves difficult in TQ to define any field analogous to the Hamiltonian fields since, given $f' \in \mathscr{F}(TQ)$, the equation for $X \in \mathscr{X}(TQ)$

$$i_X\omega_{\mathscr{L}} = \mathrm{d}f' \tag{3.5}$$

can be resolved if and only if the condition

$$L_K f' = 0 \tag{3.6}$$

holds true for every $K \in \operatorname{Ker}\omega_{\mathscr{L}}$. However, since the motion of the system is constrained to S_2, we can limit ourselves to checking to make certain that (3.6) holds true on S_2. In this case we get

$$i_X\omega_{\mathscr{L}} \underset{S_2}{=} \mathrm{d}f' \tag{3.7}$$

$$L_K f' \underset{S_2}{=} 0 \qquad \forall K \in \operatorname{Ker}\omega_{\mathscr{L}}. \tag{3.8}$$

It must be pointed out that (3.8) implies that f' must be at least weakly projectable [13], that is that a function $f \in \mathscr{F}(T^*Q)$ exists such that

$$f' \underset{S_2}{=} F\mathscr{L}^*f. \tag{3.9}$$

Using the well known fact [11] that $F\mathscr{L}_*K_{\mu_0} = X_{\phi^{(0)}_{\mu_0}}$, recalling that $F\mathscr{L}(S_2) = M_1$ and replacing (3.9) in (3.8), the latter becomes

$$0 \underset{M_1}{=} \{\phi^{(0)}_{\mu_0}, f\} \qquad \mu_0 = 1, m_1. \tag{3.10}$$

On the other hand, as has already been done for G_0 in (2.39), it is always possible to choose f so that

$$0 \underset{M_0}{=} \{\phi^{(0)}_{\mu'_0}, f\} \qquad \mu'_0 = 1, m - m_1 \tag{3.11}$$

finally obtaining

$$0 \underset{M_1}{=} \{\phi^{(0)}_{\mu}, f\} \qquad \mu = 1, m. \tag{3.12}$$

Since (3.12) is a condition in T^*Q, an attempt can be made to look for a solution of (3.7) among the $F\mathscr{L}$-projectable vector fields; that is in the set designated as $\mathscr{X}(TQ)_{F\mathscr{L}}$ by [13]. We must recall that $X \in \mathscr{X}(TQ)_{F\mathscr{L}}$ if and only if a field $X' \in \mathscr{X}(T^*Q)$ exists such that

$$L_X F\mathscr{L}^* f = F\mathscr{L}^*(L_{X'} f) \qquad \forall f \in \mathscr{F}(T^*Q) \tag{3.13}$$

$X_{(f)}$ will be used to denote the vector field which satisfies

$$X_{(f)} \in \mathscr{X}(TQ)_{F\mathscr{L}} \tag{3.14}$$

$$S(X_{(f)}) = \left(F\mathscr{L}^* \frac{\partial f}{\partial p}\right) \frac{\partial}{\partial u}. \tag{3.15}$$

A particular choice of $X_{(f)}$, which makes it possible to remove the indetermination due to elements of Ker $F\mathscr{L}_*$, is given by imposing the property

$$L_{X_{(f)}} v^{\mu} = 0 \qquad \mu = 1, m. \tag{3.16}$$

In appendix A a local expression of $X_{(f)}$ is given which satisfies (3.16).

Definition 3.1. A field $X_{(f)}$ is called *weakly Hamiltonian* if

$$F\mathscr{L}_* X_{(f)} \underset{M_1}{=} X_f. \tag{3.17}$$

As can be seen in appendix A, if (3.12) is satisfied, $X_{(f)}$ is a weakly Hamiltonian field which resolves (3.7), having taken (3.9) into account. Moreover, if f is first class on M_1, $X_{(f)}$ is tangent to the dynamical constraints. In fact,

$$L_{X_{(f)}} F\mathscr{L}^* \phi^{(1)}_{\mu_0} = F\mathscr{L}^*(L_{F\mathscr{L}_* X_{(f)}} \phi^{(1)}_{\mu_0}) \tag{3.18}$$

and since, from (3.17) we get

$$L_{F\mathscr{L}_* X_{(f)}} \phi^{(1)}_{\mu_0} \underset{M_1}{=} \{\phi^{(1)}_{\mu_0}, f\} \underset{M_1}{=} 0 \tag{3.19}$$

the pullback by $F\mathscr{L}$ of the Poisson brackets (3.19) is equal to zero on $F\mathscr{L}^{-1}(M_1)$.

For example: the fields K_{μ_0} in (2.24) are weakly Hamiltonian (and satisfy (3.16)), however, between them, only the fields K_{μ_1} are tangent to $F\mathscr{L}^{-1}(M_1)$.

Now, moving on to the problem of tangency of fields of $\mathscr{X}(TQ)$ to the submanifolds, first of all based on the properties (2.25), (2.27) and (3.16), it is easy to see that, for each assigned $X_{(f)} \in \mathscr{X}(TQ)_{F\mathscr{L}}$, the field

$$\bar{X}_{(f)} = X_{(f)} + F\mathscr{L}^*\{\lambda^{\mu'_0}, f\} K^{v}_{\mu'_0} + F\mathscr{L}^*\{\lambda^{\mu'_1}, f\} K^{v}_{\mu'_1} \tag{3.20}$$

is tangent to the submanifold defined by the non-$F\mathscr{L}$-projectable constraints (2.15) and (2.18). In other words, if $X_{(f)}$ were not tangent it becomes so through the 'correction' in (3.20) constituted of fields belonging to $V(\text{Ker}\,\omega_{\mathscr{L}})$ which are not tangent to the constraints expressed by (2.15) and (2.18). Obviously if $X_{(f)}$ is weakly Hamiltonian, so too is $\bar{X}_{(f)}$.

By means of what has been seen above in (3.19), we are now finally able to assert that $\bar{X}_{(f)}$ is tangent to S_2 if f is first class on M_1. Thus one has the noteworthy result that a vector field $\bar{X}_{(f)} \in \mathscr{X}(TQ)$ tangent to S_2 exists corresponding to a vector field $X_f \in \mathscr{X}(T^*Q)$ tangent to M_1. This bond proves quite important to the present approach to the Noether theorem.

Another useful expression, which holds true on S_2 and will be employed here, is that of the Newtonoid vector field associated with a weakly Hamiltonian field $X_{(f)}$

$$X_{(f)}(\Gamma) = X_{(f)} + S[\Gamma, X_{(f)}] \tag{3.21}$$

where Γ is a second-order solution of (1.5) on S_1 and it is tangent to S_2. In order to compute the second n components of $X_{(f)}(\Gamma)$ (i.e. $L_\Gamma(F\mathscr{L}^*\partial f/\partial p)$) one may use the known property [14]

$$L_\Gamma F\mathscr{L}^* g \underset{S_1}{=} Kg \qquad \forall g \in \mathscr{F}(T^*Q) \tag{3.22}$$

together with (2.16), (2.19) and (3.12). Thus, in just a few steps we can obtain

$$X_{(f)}(\Gamma) \underset{S_2}{=} \bar{X}_{(f)} + \left(F\mathscr{L}^* \frac{\partial}{\partial p} \{f, H^{(2)}\} + v^{\mu_1} F\mathscr{L}^* \frac{\partial}{\partial p} \{f, \phi^{(0)}_{\mu_1}\} \right) \frac{\partial}{\partial u} \tag{3.23}$$

with $H^{(2)}$ given by (2.10).

In the following sections the problem of the tangency of fields $SX_{(f)} \in V(TQ)$ to the constraint submanifolds will be addressed. In this regard a general expression has been drawn. Considering the basic relationships

$$\frac{\partial}{\partial q} F\mathscr{L}^* = F\mathscr{L}^* \frac{\partial}{\partial q} + \frac{\partial p}{\partial q} F\mathscr{L}^* \frac{\partial}{\partial p} \tag{3.24}$$

$$\frac{\partial}{\partial u} F\mathscr{L}^* = \frac{\partial p}{\partial u} F\mathscr{L}^* \frac{\partial}{\partial p} \tag{3.25}$$

and differentiating the alternative expression of Kf (see [12])

$$Kf = uF\mathscr{L}^* \frac{\partial f}{\partial q} + \frac{\partial \mathscr{L}}{\partial q} F\mathscr{L}^* \frac{\partial f}{\partial p} \tag{3.26}$$

the following is obtained for each $f, g \in \mathscr{F}(T^*Q)$

$$L_{SX_{(g)}} Kf = F\mathscr{L}^*\{f, g\} + L_{X_{(f)}(\Gamma)} F\mathscr{L}^* g + \chi^{(1)}_\mu \frac{\partial v^\mu}{\partial u} \left(F\mathscr{L}^* \frac{\partial^2 f}{\partial p \partial p} \right) \frac{\partial}{\partial u} F\mathscr{L}^* g. \tag{3.27}$$

In the previous section it was seen that the constant of motion $G \in \mathscr{F}(TQ)$ may be associated with a constant of motion $G_0 \in \mathscr{F}(T^*Q)$ which is first class on M_1. It can now be asserted that a weakly-Hamiltonain field $\bar{X}_{(G_0)}$ is associated with G and that this field is tangent to S_2. In the next section it will be necessary to know the role (2.28)-(2.34) play in the application of operator K to the function G_0. Using the well known property

$$(i_{\Gamma_0}\omega_{\mathscr{L}} - \mathrm{d}E) \circ S = 0 \qquad \forall \Gamma_0 \colon S(\Gamma_0) = \Delta \tag{3.28}$$

one easily obtains an intrinsic version of expression (15) given in [14]: for each $f \in \mathscr{F}(T^*Q)$ the following holds true

$$Kf = L_\Gamma F\mathscr{L}^* f + iX_{(f)}(i_\Gamma \omega_{\mathscr{L}} - \mathrm{d}E) \tag{3.29}$$

where Γ is a second-order field which resolves (1.5) on S_1 but which is not necessarily tangent. The equivalence of the two expressions can be demonstrated by taking Γ as in [14], thus obtaining

$$i_\Gamma \omega_{\mathscr{L}} - \mathrm{d}E = (\chi_\mu^{(1)}\, \mathrm{d}v^\mu) \circ S \tag{3.30}$$

and using this expression in (3.29).

It can be seen that for (3.29) to hold true it is sufficient to take any field which satisfies (3.15) rather than $X_{(f)}$. In fact, for (3.28) the vertical part of the field may be arbitrary. Indeed, by replacing any field X, in (3.29), for which

$$S(X) = \left(F\mathscr{L}^* \frac{\partial f}{\partial p} + a^\mu F\mathscr{L}^* \frac{\partial \phi_\mu^{(0)}}{\partial p} \right) \frac{\partial}{\partial u} \tag{3.31}$$

with arbitrary functions a^μ, (3.29) becomes

$$Kf = L_\Gamma F\mathscr{L}^* f + i_X(i_\Gamma \omega_{\mathscr{L}} - \mathrm{d}E) - a^\mu i_{Z_\mu}(i_\Gamma \omega_{\mathscr{L}} - \mathrm{d}E) \tag{3.32}$$

where the fields Z_μ are such that $SZ_\mu = K_\mu^v$. On the other hand, from (2.30) and (2.31) it immediately follows that the Newtonoid field $X(\Gamma)$ verifies (3.31) where G_0 takes the place of f. Obviously, in this case, the a^μ are fixed and, in general, non-$F\mathscr{L}$-projectable functions. Therefore the following holds true

$$KG_0 = L_\Gamma F\mathscr{L}^* G_0 + i_{X(\Gamma)}(i_\Gamma \omega_{\mathscr{L}} - \mathrm{d}E) - a^\mu i_{Z_\mu}(i_\Gamma \omega_{\mathscr{L}} - \mathrm{d}E). \tag{3.33}$$

By replacing (2.33) in (3.33) and keeping in mind that (see [20])

$$\chi_\mu^{(1)} = i_{Z_\mu}(i_{\Gamma_0} \omega_{\mathscr{L}} - \mathrm{d}E) \qquad \forall \Gamma_0\colon S(\Gamma_0) = \Delta;\ \mu = 1, m \tag{3.34}$$

the following is finally obtained

$$KG_0 = -a^\mu \chi_\mu^{(1)}. \tag{3.35}$$

4. From Noether's symmetries to dynamical symmetries

In the case of regular Lagrangian systems the condition

$$[X(\Gamma), \Gamma] = 0 \tag{4.1}$$

expresses the invariance of the equations of motion under the infinitesimal transformation generated by $X(\Gamma)$. If the Lagrange function is degenerate, the concept of dynamical symmetry must be generalized for two reasons because the motion is confined to the submanifold S_2 and, furthermore, because not only one dynamics exists; rather there is an entire equivalence class of Γ:

(*a*) $X(\Gamma)$ must be tangent to S_2: so integral curves of Γ lying on S_2 are mapped onto other integral curves which also lie on S_2;

(*b*) a set of m_2 functions $b^{\mu_1} \in \mathscr{F}(TQ)$ must exist such that

$$[X(\Gamma), \Gamma] \underset{S_2}{=} b^{\mu_1} K_{\mu_1}^v \tag{4.2}$$

so that $X(\Gamma)$ transforms Γ into the equivalent dynamics $\Gamma' \underset{S_2}{=} \Gamma - \varepsilon b^{\mu_1} K_{\mu_1}^v$ (ε being an infinitesimal parameter).

If (*a*) and (*b*) hold true we say that $X(\Gamma)$ is a dynamical symmetry transformation (DST). This definition is complementary to that of dynamical symmetry in T^*Q [16, 21]: as a matter of fact, a DST transforms integral curves of the dynamics into gauge-equivalent curves.

First of all we wish to demonstrate that if $X \in \mathscr{X}(TQ)$ satisfies (1.3), $X(\Gamma)$ is always tangent to S_2: to do so some properties derived from (1.3) will be analysed.

From (1.2), that is from the definition of the Newtonoid field, and from

$$SX(\Gamma) = \left(F\mathscr{L}^* \frac{\partial G_0}{\partial p} + a^\mu F\mathscr{L}^* \frac{\partial \phi_\mu^{(0)}}{\partial p} \right) \frac{\partial}{\partial u} \tag{4.3}$$

(similar to (3.31)) which provides the first n components of $X(\Gamma)$, we get

$$X(\Gamma) = X_{(G_0)}(\Gamma) + a^\mu Z_\mu(\Gamma) + (L_\Gamma a^\mu) K_\mu^v. \tag{4.4}$$

Now we shall demonstrate the following.

Lemma 4.1. If $X \in \mathscr{X}(TQ)$ satisfies (1.3) then

$$X(\Gamma) \underset{S_2}{=} X_{(G_0)}(\Gamma) + a^{\mu_1} Z_{\mu_1}(\Gamma) + (L_\Gamma a^{\mu_1}) K_{\mu_1}^v. \tag{4.5}$$

Proof. We shall first demonstrate that

$$a^{\mu_0'} \underset{S_1}{=} 0 \qquad \mu_0' = 1, m - m_1. \tag{4.6}$$

In fact, placing $g = \phi_{\mu_0'}^{(0)}$ and $f = G_0$ in (3.27), from (2.39) we get

$$L_{K_{\mu_0'}^v}(KG_0) = 0 \tag{4.7}$$

and differentiating (3.35) with respect to $K_{\mu_0'}^v$ we get

$$L_{K_{\mu_0'}^v}(KG_0) = -\chi_\nu^{(1)} L_{K_{\mu_0'}^v} a^\nu - a^{\nu_0'} F\mathscr{L}^* \{\phi_{\nu_0'}^{(0)}, \phi_{\mu_0'}^{(0)}\}. \tag{4.8}$$

Equations (4.7) and (4.8) imply

$$a^{\nu_0'} FL^* \{\phi_{\nu_0'}^{(0)}, \phi_{\mu_0'}^{(0)}\} \underset{S_1}{=} 0 \qquad \mu_0' = 1, m - m_1 \tag{4.9}$$

which, because of (2.4), proves (4.6).

Now we shall demonstrate that

$$a^{\mu_1'} \underset{S_2}{=} 0 \qquad \mu_1' = 1, m_1 - m_2. \tag{4.10}$$

First of all, using (3.22), it can be observed that

$$\begin{aligned} L_\Gamma(KG_0) &= L_\Gamma(F\mathscr{L}^* \{G_0, H\} + v^\mu F\mathscr{L}^* \{G_0, \phi_\mu^{(0)}\}) \\ &\underset{S_1}{=} K\{G_0, H\} + v^\mu K\{G_0, \phi_\mu^{(0)}\} \end{aligned} \tag{4.11}$$

where the fact that G_0 is a first-class function on M_1 is used. On the other hand, using (3.35) and (4.6) one obtains

$$L_\Gamma(KG_0) \underset{S_1}{=} -\alpha^{\nu_1'} \chi_{\nu_1'}^{(2)}. \tag{4.12}$$

At this point we can differentiate both (4.11) and (4.12) with respect to fields $K_{\mu_1'}^v$, which are tangent to S_1. Using (3.27) once more and again recalling that G_0 is a first-class function, one respectively obtains

$$\begin{aligned} &L_{K_{\mu_1'}^v} L_\Gamma(KG_0) \\ &\quad \underset{S_1}{=} F\mathscr{L}^* \{\{G_0, \phi_{\mu_1'}^{(0)}\}, H\} + v^\mu F\mathscr{L}^* \{\{G_0, \phi_{\mu_1'}^{(0)}\}, \phi_\mu^{(0)}\} \\ &\qquad + F\mathscr{L}^* \{\{G_0, H\}, \phi_{\mu_1'}^{(0)}\} + v^\mu F\mathscr{L}^* \{\{G_0, \phi_\mu^{(0)}\} \phi_{\mu_1'}^{(0)}\} \end{aligned} \tag{4.13}$$

$$L_{K_{\mu_1'}^v} L_\Gamma(KG_0) \underset{S_1}{=} -\chi_{\nu_1'}^{(2)} L_{K_{\mu_1'}^v} a^{\nu_1'} - a^{\nu_1'} F\mathscr{L}^* \{\phi_{\nu_1'}^{(1)}, \phi_{\mu_1'}^{(0)}\}. \tag{4.14}$$

Applying the Jacoby identity, along with (2.16), (2.19) and (2.10) to (4.13) and subsequently equating (4.13) and (4.14) on S_2 we obtain

$$0 \underset{S_2}{=} a^{\nu'_1} F\mathscr{L}^*\{\phi^{(1)}_{\nu'_1}, \phi^{(0)}_{\mu'_1}\} \qquad \mu'_1 = 1, m_1 - m_2. \tag{4.15}$$

From (4.15) and (2.9) the result (4.10) follows. Thus the lemma has been proved. □

We shall now write the fields $X_{(G_0)}(\Gamma)$ and $K_{\mu_1}(\Gamma)$ as in equation (3.23)

$$X_{(G_0)}(\Gamma) \underset{S_2}{=} \bar{X}_{(G_0)} + \left(F\mathscr{L}^* \frac{\partial}{\partial p}\{G_0, H^{(2)}\} + v^{\mu_1} F\mathscr{L}^* \frac{\partial}{\partial p}\{G_0, \phi^{(0)}_{\mu_1}\}\right)\frac{\partial}{\partial u} \tag{4.16}$$

$$K_{\mu_1}(\Gamma) \underset{S_2}{=} \bar{X}_{(\phi^{(0)}_{\mu_1})} + \left(F\mathscr{L}^* \frac{\partial}{\partial p}\{\phi^{(0)}_{\mu_1}, H^{(2)}\} + v^{\nu_1} F\mathscr{L}^* \frac{\partial}{\partial p}\{\phi^{(0)}_{\mu_1}, \phi^{(0)}_{\nu_1}\}\right)\frac{\partial}{\partial u}. \tag{4.17}$$

As was seen in the previous section, as G_0 and $\phi^{(0)}_{\mu_1}$ are first-class functions on M_1, the fields $\bar{X}_{(G_0)}$ and $\bar{X}_{(\phi^{(0)}_{\mu_1})}$ are weakly-Hamiltonian fields, tangent to S_2. Since the functions $\{\phi^{(0)}_{\mu_1}, \phi^{(0)}_{\nu_1}\}$, on the grounds of known involutive properties, are first-class functions the vector field

$$SX_{(\{\phi^{(0)}_{\mu_1}, \phi^{(0)}_{\nu_1}\})} = F\mathscr{L}^* \frac{\partial}{\partial p}\{\phi^{(0)}_{\mu_1}, \phi^{(0)}_{\nu_1}\}\frac{\partial}{\partial u} \tag{4.18}$$

is also tangent to S_2 as is $K^v_{\mu_1}$. With the aid of these considerations, if (4.16) and (4.17) are replaced in (4.5) one can immediately see that $X(\Gamma)$ is tangent to S_2 if and only if the field

$$V = \left(F\mathscr{L}^* \frac{\partial}{\partial p}\{G_0, H^{(2)}\} + v^{\mu_1} F\mathscr{L}^* \frac{\partial}{\partial p}\{G_0, \phi^{(0)}_{\mu_1}\} + a^{\mu_1} F\mathscr{L}^* \frac{\partial}{\partial p}\{\phi^{(0)}_{\mu_1}, H^{(2)}\}\right)\frac{\partial}{\partial u} \tag{4.19}$$

is likewise tangent to S_2. This is exactly what happens as will now be proved in order to demonstrate the following.

Proposition 4.1. Given a symmetry for the Lagrangian $X \in \mathscr{X}(TQ)$, if Γ is the second-order dynamics of the system, tangent to the final constraint submanifold, then $X(\Gamma)$ is tangent to that submanifold.

Proof. Let us begin by recalling that the following holds true

$$i_{X_{(G_0)}}\omega_{\mathscr{L}} \underset{S_2}{=} \mathrm{d}F\mathscr{L}^* G_0. \tag{4.20}$$

Since the analogous result (2.31) also holds true, $\omega_{\mathscr{L}}$ can be contracted with the field (4.5) thus obtaining

$$i_V\omega_{\mathscr{L}} \underset{S_2}{=} 0. \tag{4.21}$$

From (4.21), using (3.25), one finds

$$0 \underset{S_2}{=} \frac{\partial}{\partial u}(F\mathscr{L}^*\{G_0, H^{(2)}\}) + v^{\mu_1}\frac{\partial}{\partial u}(F\mathscr{L}^*\{G_0, \phi^{(0)}_{\mu_1}\}) + a^{\mu_1}\frac{\partial}{\partial u}(F\mathscr{L}^*\{\phi^{(0)}_{\mu_1}, H^{(2)}\})$$

$$\underset{S_2}{=} \frac{\partial}{\partial u}(F\mathscr{L}^*\{G_0, H^{(2)}\} + v^{\mu_1} F\mathscr{L}^*\{G_0, \phi^{(0)}_{\mu_1}\} + a^{\mu_1} F\mathscr{L}^*\{\phi^{(0)}_{\mu_1}, H^{(2)}\}) \tag{4.22}$$

where the last passage is made possible by (2.36) and (2.37) and by the fact that there are no tertiary Hamiltonian constraints, therefore

$$0 \underset{S_2}{=} F\mathscr{L}^*\{G_0, H^{(2)}\} \underset{S_2}{=} F\mathscr{L}^*\{G_0, \phi^{(0)}_{\mu_1}\} \underset{S_2}{=} F\mathscr{L}^*\{\phi^{(0)}_{\mu_1}, H^{(2)}\}. \tag{4.23}$$

In appendix B it is shown that, if there are no constraints depending essentially on the q^{i}'s, given a function $f \in \mathscr{F}(TQ)$ such that

$$f \underset{S_2}{=} 0 \tag{4.24}$$

$$\frac{\partial f}{\partial u} \underset{S_2}{=} 0 \tag{4.25}$$

it necessarily follows that

$$\frac{\partial f}{\partial q} \underset{S_2}{=} 0. \tag{4.26}$$

This means that, if (4.24) and (4.25) hold true, then

$$f \underset{S_2}{\equiv} 0 \tag{4.27}$$

where the symbol $\equiv$ stands for 'strongly equal to' [9]. Therefore, from (4.22) and (4.23) one can draw

$$F\mathscr{L}^*\{G_0, H^{(2)}\} + v^{\mu_1} F\mathscr{L}^*\{G_0, \phi^{(0)}_{\mu_1}\} + a^{\mu_1} F\mathscr{L}^*\{\phi^{(0)}_{\mu_1}, H^{(2)}\} \underset{S_2}{\equiv} 0. \tag{4.28}$$

Due to the previously mentioned involutive properties, all the PBs appearing in (4.23) are first-class functions on M_1.

If in (3.27) either g or f are first-class functions and the other is a constraint it follows that

$$L_{SX_{(g)}} Kf \underset{S_1}{=} L_{X_{(f)}(\Gamma)} F\mathscr{L}^* g. \tag{4.29}$$

Property (4.29) can, with the help of (4.23), be used to calculate the Lie derivative of each Lagrangian constraint with respect to V. The result is

$$L_V K\phi \underset{S_2}{=} L_{X_{(\phi)}(\Gamma)}(F\mathscr{L}^*\{G_0, H^{(2)}\} + v^{\mu_1}\{G_0, \phi^{(0)}_{\mu_1}\} + a^{\mu_1} F\mathscr{L}^*\{\phi^{(0)}_{\mu_1}, H^{(2)}\}). \tag{4.30}$$

In (4.30) all the Lagrangian constraints can be obtained by varying ϕ between all the Hamiltonian constraints. Finally one can see that the right-hand side of (4.30) is equal to zero on S_2 since (4.28) holds true. Thus proposition 4.1 has been proved. □

At this point we can finally prove:

Proposition 4.2. Given a Lagrangian symmetry $X \in \mathscr{X}(TQ)$, the Newtonoid vector field $X(\Gamma)$ is a DST; i.e. the dynamics of the constrained system is invariant under the infinitesimal transformation generated by $X(\Gamma)$.

Proof. Let us compute the following contraction

$$i_{[X(\Gamma),\Gamma]}\omega_{\mathscr{L}} = L_{X(\Gamma)} i_\Gamma \omega_{\mathscr{L}} - i_\Gamma L_{X(\Gamma)} \omega_{\mathscr{L}} \underset{S_2}{=} L_{X(\Gamma)}\, \mathrm{d}E. \tag{4.31}$$

The last passage in (4.31) is made possible only because the above proposition holds true. Therefore, based on (2.32), the second member of (4.31) is equal to zero. Thus, from (1.1) one obtains (4.2), $\omega_{\mathscr{L}}$ being of constant rank. □

5. The converse Noether theorem

Let us assume we have a constant of motion $G \in \mathscr{F}(TQ)$ for a degenerate Lagrangian system, i.e.

$$L_\Gamma G \underset{S_2}{=} 0. \tag{5.1}$$

Since, as was mentioned in section 2, the dynamics Γ tangent to S_2 is determined except for any linear combination of fields $K^v_{\mu_1}$, with $\mu_1 = 1, m_2$, we get

$$L_{K^v_{\mu_1}} G \underset{S_2}{=} 0 \qquad \mu_1 = 1, m_2. \tag{5.2}$$

Therefore a function $G_0 \in \mathscr{F}(T^*Q)$ exists such that

$$F\mathscr{L}^* G_0 \underset{S_2}{=} G. \tag{5.3}$$

As in the case of the direct theorem, we shall demonstrate that is possible to choose G_0 so that it is first class over M_1. For every G_0 which verifies (5.3) the following properties hold true

$$\{G_0, \phi^{(0)}_{\nu_1}\} \underset{M_1}{=} 0 \qquad \nu_1 = 1, m_2 \tag{5.4}$$

$$\{G_0, H^{(2)}\} \underset{M_1}{=} 0. \tag{5.5}$$

In fact, using (5.1) and (3.22), together with (2.16), (2.19) and (2.10), from relation (5.3) it follows that

$$0 \underset{S_2}{=} KG_0 \underset{S_2}{=} F\mathscr{L}^*\{G_0, H^{(2)}\} + v^{\mu_1} F\mathscr{L}^*\{G_0, \phi^{(0)}_{\mu_1}\}. \tag{5.6}$$

Since the fields $K^v_{\nu_1}$ are tangent to S_2, (5.6) can be differentiated with respect to each of them with the help of (2.25). Recalling that $F\mathscr{L}(S_2) = M_1$, (5.4) are obtained and, as a consequence, (5.5) is also obtained.

The arbitrariness present in G_0 is even greater in the converse theorem than in the direct theorem and it is through a partial removal of this arbitrariness that one can choose a G_0 of first class. In fact, given a G_0' that also satisfies equation (5.3),

$$G_0 = G_0' + w^\mu \phi^{(0)}_\mu + u^{\mu_0} \phi^{(1)}_{\mu_0} \tag{5.7}$$

satisfies (5.3) for every choice of the functions w^μ, $u^{\mu_0} \in \mathscr{F}(T^*Q)$. Keeping equation (2.9) in mind, it becomes clear that the functions $u^{\mu_1'}$ can be determined in such a manner that

$$\{G_0, \phi^{(0)}_{\nu_1'}\} \underset{M_1}{=} 0 \qquad \nu_1' = 1, m_1 - m_2. \tag{5.8}$$

With the aid of (5.8), (5.4) and (5.5) become, respectively, equal to (2.36) and (2.37). If the latter two hold true, a first-class G_0 can be obtained by fixing $w^{\mu_0'}$ and $w^{\mu_1'}$ by means of the same procedure employed in section 2. Referring, from this point on, just to this function G_0 we shall now prove the following.

Lemma 5.1. Let $G_0 \in \mathscr{F}(T^*Q)$ be first-class on M_1 and $\{G_0, H^{(2)}\} \underset{M_1}{=} 0$. It is then always possible to find a linear combination of dynamical constraints such that

$$KG_0 = C^{\mu_0} \chi^{(1)}_{\mu_0} \tag{5.9}$$

$$KG_0 \underset{S_2}{\equiv} C^{\mu_1} \chi^{(1)}_{\mu_1}. \tag{5.10}$$

Proof. Developing (2.12) with the aid of (2.10), and recalling that $F\mathscr{L}^*\{G_0, \phi^{(0)}_{\mu'_0}\}=0$, $\mu'_0=1, m-m_1$, because of (2.39), we obtain

$$KG_0 = F\mathscr{L}^*\{G_0, H^{(2)}\} + v^{\mu_1}F\mathscr{L}^*\{G_0, \phi^{(0)}_{\mu_1}\} + (v^{\mu'_1} - F\mathscr{L}^*\lambda^{\mu'_1})F\mathscr{L}^*\{G_0, \phi^{(0)}_{\mu'_1}\}. \tag{5.11}$$

By hypothesis all the PBs contained in (5.11) vanish on M_1. This implies that some functions α^{μ_0}, $\beta^{\mu_0}_{\mu_1}$, $\gamma^{\mu_0}_{\mu'_1}$, η^{μ}, $\rho^{\mu}_{\mu_1}$, $\sigma^{\mu}_{\mu'_1} \in \mathscr{F}(T^*Q)$ exist such that

$$\{G_0, H^{(2)}\} = \eta^{\mu}\phi^{(0)}_{\mu} + \alpha^{\mu_0}\phi^{(1)}_{\mu_0} \tag{5.12}$$

$$\{G_0, \phi^{(0)}_{\mu_1}\} = \rho^{\mu}_{\mu_1}\phi^{(0)}_{\mu} + \beta^{\mu_0}_{\mu_1}\phi^{(1)}_{\mu_0} \qquad \mu_1 = 1, m_2 \tag{5.13}$$

$$\{G_0, \phi^{(0)}_{\mu'_1}\} = \sigma^{\mu}_{\mu'_1}\phi^{(0)}_{\mu} + \gamma^{\mu_0}_{\mu'_1}\phi^{(1)}_{\mu_0} \qquad \mu'_1 = 1, m_1 - m_2. \tag{5.14}$$

Performing the pullback of these relations one sees that

$$F\mathscr{L}^*\{G_0, H^{(2)}\} = (F\mathscr{L}^*\alpha^{\mu_0})\chi^{(1)}_{\mu_0} \tag{5.15}$$

$$v^{\mu_1}F\mathscr{L}^*\{G_0, \phi^{(0)}_{\mu_1}\} = (v^{\mu_1}F\mathscr{L}^*\beta^{\mu_0}_{\mu_1})\chi^{(1)}_{\mu_0} \tag{5.16}$$

$$(v^{\mu'_1} - F\mathscr{L}^*\lambda^{\mu'_1})F\mathscr{L}^*\{G_0, \phi^{(0)}_{\mu'_1}\} = ((v^{\mu'_1} - F\mathscr{L}^*\lambda^{\mu'_1})F\mathscr{L}^*\gamma^{\mu_0}_{\mu'_1})\chi^{(1)}_{\mu_0}. \tag{5.17}$$

Replacing (5.15), (5.16) and (5.17) in (5.11) one can see that (5.9) is satisfied by taking

$$C^{\mu_0} = F\mathscr{L}^*\alpha^{\mu_0} + v^{\mu_1}F\mathscr{L}^*\beta^{\mu_0}_{\mu_1} + (v^{\mu'_1} - F\mathscr{L}^*\lambda^{\mu'_1})F\mathscr{L}^*\gamma^{\mu_0}_{\mu'_1}. \tag{5.18}$$

Since the PBs in (5.12) and (5.13) are first-class functions on M_1, the functions $\eta^{\mu'_0}$, $\eta^{\mu'_1}$, $\alpha^{\mu'_1}$, $\rho^{\mu'_0}_{\mu_1}$, $\rho^{\mu'_1}_{\mu_1}$, $\beta^{\mu'_1}_{\mu_1}$ must vanish on M_1. Therefore

$$C^{\mu_1} \underset{S_2}{=} F\mathscr{L}^*\alpha^{\mu_1} + v^{\nu_1}F\mathscr{L}^*\beta^{\mu_1}_{\nu_1} \tag{5.19}$$

$$C^{\mu'_1} \underset{S_2}{=} 0 \tag{5.20}$$

hold true.

The result (4.10) follows immediately from (5.20) since the product of two weakly vanishing functions is a strongly vanishing function. □

The subsequent step in the demonstration of the converse Noether theorem links a DST to the constant of motion. First of all we see that the vector field $X_{(G_0)}(\Gamma)$, as it appears in (4.16), is not necessarily tangent to S_2. Since this is an essential property of a DST we must build up a suitable Newtonoid field $X(\Gamma)$ having this property. To this purpose we observe that since G_0 is first class on M_1, the field $X_{(G_0)}$ is a weakly-Hamiltonian field and thus resolves

$$i_{X_{(G_0)}}\omega_{\mathscr{L}} \underset{S_2}{=} \mathrm{d}G. \tag{5.21}$$

Equation (5.21) is likewise satisfied by the field

$$X = X_{(G_0)} + a^{\mu_0}K_{\mu_0} \tag{5.22}$$

where the functions $a^{\mu_0} \in \mathscr{F}(TQ)$ are completely arbitrary. Therefore we can demonstrate the following.

Proposition 5.1. If $G \in \mathscr{F}(TQ)$ is a constant of motion on the final constraint submanifold S_2, the equation

$$i_X \omega_{\mathscr{L}} \underset{S_2}{=} \mathrm{d}G \tag{5.23}$$

allows for at least one solution $X \in \mathscr{X}(TQ)$ such that the field

$$X(\Gamma) = X + S[\Gamma, X] \tag{5.24}$$

is tangent to S_2 and satisfies

$$i_{X(\Gamma)} \omega_{\mathscr{L}} \underset{S_2}{=} \mathrm{d}G. \tag{5.25}$$

Proof. It is sufficient to take, in (5.22),

$$a^{\mu_0} = -C^{\mu_0} \tag{5.26}$$

where C^{μ_0} is given by (5.18) of the lemma, and use the same arguments given in the previous section to verify that the field

$$X(\Gamma) \underset{S_2}{=} X_{(G_0)}(\Gamma) + a^{\mu_1} K_{\mu_1}(\Gamma) + (L_\Gamma a^{\mu_1}) K^{v}_{\mu_1} \tag{5.27}$$

satisfies equation (5.25) and is tangent to S_2.

As in the direct theorem it is sufficient to show that the field V, written as in (4.19), satisfies equation (4.21) and is tangent to S_2. To see this it can be noted that from (5.10) we obtain

$$F\mathscr{L}^*\{G_0, H^{(2)}\} + v^{\mu_1} F\mathscr{L}^*\{G_0, \phi^{(0)}_{\mu_1}\} + a^{\mu_1} F\mathscr{L}^*\{\phi^{(0)}_{\mu_1}, H^{(2)}\} \underset{S_2}{\equiv} 0 \tag{5.28}$$

and thus we obtain

$$i_V \omega_{\mathscr{L}} \underset{S_2}{=} 0 \tag{5.29}$$

and

$$L_V K\phi \underset{S_2}{=} 0 \tag{5.30}$$

where ϕ is any Hamiltonian constraint. Therefore equation (5.25) is solved taking $X(\Gamma)$ as in (5.27), because of (4.16) and (5.21). The tangency of $X(\Gamma)$ to S_2 is, on the other hand, ensured by (4.16), (4.17), (4.18) and (5.30). Furthermore, as in (4.31), the following

$$i_{[X(\Gamma),\Gamma]} \omega_{\mathscr{L}} \underset{S_2}{=} 0 \tag{5.31}$$

is obtained.

Thus it has been proved that $X(\Gamma)$ is a DST for the class of equivalent vector fields represented by Γ. □

Now, in order to pass from a dynamical symmetry to a generalized Lagrangian symmetry it must first of all be shown that if we define

$$F := i_{X(\Gamma)}(\mathrm{d}\mathscr{L} \circ S) - F\mathscr{L}^* G_0 \qquad F \in \mathscr{F}(TQ) \tag{5.32}$$

we obtain

$$L_{X(\Gamma)} \mathscr{L} = L_\Gamma F. \tag{5.33}$$

To obtain this result it is enough to differentiate (5.32) with respect to Γ: in a few steps we obtain

$$L_\Gamma F = L_{X(\Gamma)}\mathscr{L} + i_{X(\Gamma)}(\mathrm{d}E - i_\Gamma \omega_{\mathscr{L}}) - L_\Gamma F\mathscr{L}^* G_0. \tag{5.34}$$

Now, taking property (3.32) into consideration and keeping in mind the fact that the functions a^{μ_0} are fixed functions in proposition 5.1 and that in (5.22)

$$a^{\mu_0'} = 0 \qquad \mu_0' = 1, m_1 - m_2 \tag{5.35}$$

we can write

$$L_{X(\Gamma)}\mathscr{L} - L_\Gamma F = KG_0 + a^{\mu_0} i_{K_{\mu_0}}(i_\Gamma \omega_{\mathscr{L}} - \mathrm{d}E). \tag{5.36}$$

The RHS of (5.36) vanishes because of (5.9) and so (5.33) follows.

Thus we can enunciate the following proposition which constitutes the presymplectic version of the converse Noether theorem.

Proposition 5.2. Let $G \in \mathscr{F}(TQ)$ and $\Gamma \in \mathscr{X}(TQ)$ be any second-order dynamics which satisfies the equation $i_\Gamma \omega_{\mathscr{L}} = \mathrm{d}E$ and which is tangent to the final constraint submanifold S_2. If the conservation law

$$L_\Gamma G \underset{S_2}{=} 0 \tag{5.37}$$

holds true, then it is always possible to find a vector field $X \in \mathscr{X}(TQ)$ such that:

(a) $X(\Gamma)$ is a DST;

(b) X is a generalized symmetry of the Lagrange function.

Proof. Keeping the preceeding results in mind, what is left to do is demonstrate that

$$L_{X(\Gamma_0)}\mathscr{L} = L_{\Gamma_0} F \qquad \forall \Gamma_0 \in \mathscr{X}(TQ)\colon S(\Gamma_0) = \Delta \tag{5.38}$$

where F is defined in (5.32) and G_0 in (5.3).

For every second-order Γ_0 written as

$$\Gamma_0 = \Gamma - V \qquad V \in V(TQ) \tag{5.39}$$

it holds true that

$$X(\Gamma_0 + V) = X(\Gamma_0) + S[V, X] \tag{5.40}$$

and, moreover, (5.33) becomes

$$L_{X(\Gamma_0)}\mathscr{L} + L_{S[V,X]}\mathscr{L} - L_{\Gamma_0}F - L_V F = 0. \tag{5.41}$$

On the other hand, using (5.32) one has $\forall V \in V(TQ)$

$$L_{S[V,X]}\mathscr{L} = L_V F + L_V F\mathscr{L}^* G_0 - i_V i_X \omega_{\mathscr{L}}. \tag{5.42}$$

From the fact that (see [6]) all vertical subspaces are Lagrangian for $\omega_{\mathscr{L}}$, it follows that

$$i_V i_{S[\Gamma_0, X]}\omega_{\mathscr{L}} = 0 \qquad \forall V \in V(TQ). \tag{5.43}$$

Adding such a null term to the RHS of equation (5.42) and keeping in mind that

$$(F\mathscr{L}^* \,\mathrm{d}G_0 - i_{X(\Gamma_0)}\omega_{\mathscr{L}}) \circ S = 0 \tag{5.44}$$

one obtains $\forall V \in V(TQ)$

$$L_{S[V,X]}\mathscr{L} - L_V F = i_V(F\mathscr{L}^* \,\mathrm{d}G_0 - i_{X(\Gamma_0)}\omega_{\mathscr{L}}) = 0 \tag{5.45}$$

which, once replaced in (5.41), provides the statement (5.38).

6. Conclusions

In this work a generalization of the Noether theorem to degenerate Lagrangian systems has been put forward. The proof of the theorem is performed within the Lagrangian framework, under the quite general hypothesis that TQ is a presymplectic manifold. To do so it has been necessary to provide a suitable generalization of the concept of dynamical symmetry which take into account the fibre structure of TQ as well as the arbitrariness of the dynamics.

If we compare said definition with the similar one provided by [16] for analysis in T^*Q, it is easy to see that under hypotheses less general than those given here, a DST in TQ is associated with a dynamical symmetry in T^*Q (in essence this happens whenever the first components of the symmetry $X(\Gamma)$ are $F\mathscr{L}$-projectable).

Moreover, the converse theorem is demonstrated by means of an appropriate construction of the symmetry starting from the constant of motion. In this manner the structure of the proof of the Noether theorem given by [2] and based on the generalization from Newtonian transformation to Newtonoid transformation and carried out without resorting to the use of higher space TTQ has been maintained.

Appendix A

A local expression will be given for the fields denoted in the present work by $X_{(f)}$. For a given $f \in \mathscr{F}(T^*Q)$ they must satisfy (3.14) and (3.15). To this purpose a remark about Hamiltonian constraints is needed.

Writing the Legendre transformation so that the relations

$$p_\rho = \frac{\partial \mathscr{L}}{\partial u^\rho} \qquad \rho = n-m+1, n \tag{A.1}$$

express the last m moments as functions of the first $(n-m)$ and of the coordinates q, we can write

$$p_\rho = \psi_\rho(q^i, p_\alpha) \qquad i = 1, n;\ \rho = n-m+1, n;\ \alpha = 1, n-m. \tag{A.2}$$

As in [8, 9] the m primary constraints can be written as

$$\varphi_\rho = p_\rho - \psi_\rho \qquad \rho = n-m+1, n. \tag{A.3}$$

It must be recalled that the constraints $\phi_\mu^{(0)}$ introduced with (2.1) are an appropriate linear combination of (A.3) making it possible to identify the maximal set of independent first-class constraints. We shall make use of the relationship

$$\phi_\mu^{(0)} \underset{M_0}{\equiv} \frac{\partial \phi_\mu^{(0)}}{\partial p_\rho} \varphi_\rho \tag{A.4}$$

which comes down from (62) in [9].

From (3.15) we can write

$$X_{(f)} = F\mathscr{L}^* \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} + X^i \frac{\partial}{\partial u^i}. \tag{A.5}$$

Indeed, we impose condition (3.14) on the X^i's: since $X_{(f)} \in \mathscr{X}(TQ)_{F\mathscr{L}}$ if $\exists\, Y \in \mathscr{X}(T^*Q)$ such that $F\mathscr{L}_* X = Y$, there must be n functions $Y^k \in \mathscr{F}(T^*Q)$ such that

$$\frac{\partial p_k}{\partial q^i} F\mathscr{L}^* \frac{\partial f}{\partial p_i} + W_{ik} X^i = F\mathscr{L}^* Y^k \qquad k = 1, n. \tag{A.6}$$

Since the Hessian W is degenerate, the system of equations (A.6) can be solved for X^i if and only if the following condition holds true

$$F\mathscr{L}^* Y^k \frac{\partial \phi_\mu^{(0)}}{\partial p_k} - \frac{\partial p_k}{\partial q^i} F\mathscr{L}^* \frac{\partial f}{\partial p_i} \frac{\partial \phi_\mu^{(0)}}{\partial p_k} = 0 \qquad \mu = 1, m. \tag{A.7}$$

Using (A.4) one can easily see that

$$Y^k = -\frac{\partial f}{\partial q^k} - \delta_\rho^k \{\varphi_\rho, f\} \tag{A.8}$$

satisfies (A.7).

In order to resolve equation (A.6) we shall make use of the relationship of completeness (see [12])

$$W_{ik} M^{kj} = \delta_i^j - \frac{\partial v^\mu}{\partial u^i} F\mathscr{L}^* \frac{\partial \phi_\mu^{(0)}}{\partial p_j} \tag{A.9}$$

with

$$M^{kj} = F\mathscr{L}^* \frac{\partial^2 H}{\partial p_k \partial p_j} + v^\mu F\mathscr{L}^* \frac{\partial^2 \phi_\mu^{(0)}}{\partial p_k \partial p_j}. \tag{A.10}$$

We obtain

$$X^i = F\mathscr{L}^* \left\{ \frac{\partial H}{\partial p_i}, f \right\} + v^\mu F\mathscr{L}^* \left\{ \frac{\partial \phi_\mu^{(0)}}{\partial p_i}, f \right\} + M_{i\rho} F\mathscr{L}^* \{f, \varphi_\rho\} + \alpha^\nu F\mathscr{L}^* \frac{\partial \phi_\nu^{(0)}}{\partial p_i} \tag{A.11}$$

where the last term represents all arbitrariness due to the degeneration of W.

Differentiating with respect to the $X_{(f)}$ obtained in this manner the Lagrangian identity (see [14])

$$u^i = F\mathscr{L}^* \frac{\partial H}{\partial p_i} + v^\nu F\mathscr{L}^* \frac{\partial \phi_\nu^{(0)}}{\partial p_i} \tag{A.12}$$

one obtains

$$L_{X_{(f)}} v^\nu = \alpha^\nu \qquad \nu = 1, m. \tag{A.13}$$

If, to remove the arbitrariness contained in (A.11), we set

$$\alpha^\nu = 0 \qquad \nu = 1, m \tag{A.14}$$

we obtain (3.16).

Finally, equations (3.14)-(3.16) are satisfied by

$$X_{(f)} = F\mathscr{L}^* \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} + \left(F\mathscr{L}^* \left\{ \frac{\partial H}{\partial p_i}, f \right\} + v^\mu F\mathscr{L}^* \left\{ \frac{\partial \phi_\mu^{(0)}}{\partial p_i}, f \right\} \right.$$
$$\left. + M_{i\rho} F\mathscr{L}^* \{f, \varphi_\rho\} \right) \frac{\partial}{\partial u^i}. \tag{A.15}$$

As a verification, through direct calculation it is easy to show that

$$[K_\mu^v, X_{(f)}] = 0 \qquad \mu = 1, m \tag{A.16}$$

which is another way to acquire the $F\mathscr{L}$-projectability of $X_{(f)}$.

Finally, we can also write that

$$F\mathscr{L}_*X_{(f)}=\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i}-\frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}+\{f,\varphi_\rho\}\frac{\partial}{\partial p_\rho} \tag{A.17}$$

from which it is easy to see under what condition

$$F\mathscr{L}_*X_{(f)}\underset{M_0}{=}X_f \tag{A.18}$$

or

$$F\mathscr{L}_*X_{(f)}\underset{M_1}{=}X_f. \tag{A.19}$$

Appendix B

Let us assume that constraint analysis has determined a submanifold $S\subset TQ$ and that no constraint can be reduced to a condition on the q's alone: this is equivalent to ruling out the possibility that the dimensions of Q are reduced by the equations of motion.

Consequently, let us assume that we can describe S with $n+s$ coordinates, making explicit its identification mapping

$$j_s: S\hookrightarrow TQ \tag{B.1}$$

as follows

$$\begin{aligned} &q^i=q^i && i=1,n\\ &u^\beta=u^\beta && \beta=1,s\\ &u^\alpha=\psi^\alpha(q^i,u^\beta) && \alpha=s+1,n. \end{aligned} \tag{B.2}$$

It can easily be demonstrated that $\forall F\in\mathscr{F}(TQ)$ such that $j_s^*F=0$

$$j_s^*\frac{\partial F}{\partial u^\alpha}=0 \qquad (\alpha=s+1,n)\Rightarrow\begin{cases} j_s^*\dfrac{\partial F}{\partial q^i}=0 & i=1,n\\ j_s^*\dfrac{\partial F}{\partial u^\beta}=0 & \beta=1,s. \end{cases} \tag{B.3}$$

In fact, using

$$\frac{\partial'}{\partial q^i}\qquad \frac{\partial'}{\partial u^\beta}\qquad i=1,n;\ \beta=1,s \tag{B.4}$$

to indicate a basis of $\mathscr{X}(S)$ one can write

$$j_{s*}\frac{\partial'}{\partial q^i}=\frac{\partial}{\partial q^i}+\frac{\partial\psi^\alpha}{\partial q^i}\frac{\partial}{\partial u^\alpha}\qquad i=1,n \tag{B.5}$$

$$j_{s*}\frac{\partial'}{\partial u^\beta}=\frac{\partial}{\partial u^\beta}+\frac{\partial\psi^\alpha}{\partial u^\beta}\frac{\partial}{\partial u^\alpha}\qquad \beta=1,s \tag{B.6}$$

and since, if $j_s^* F = 0$, one also has

$$\frac{\partial'}{\partial q^i} j_s^* F = j_s^* L_{j_{s*}\partial'/\partial q^i} F = 0 \qquad i = 1, n \tag{B.7}$$

$$\frac{\partial'}{\partial u^\beta} j_s^* F = j_s^* L_{j_{s*}\partial'/\partial u^\beta} F = 0 \qquad \beta = 1, s \tag{B.8}$$

proposition (B.3) is at once proved.

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